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The spectrum of two quantum layers coupled by a window

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Abstract

We consider the Dirichlet Laplacian in a domain formed by two three-dimensional parallel layers having common boundary and coupled by a window. The window produces the bound states below the essential spectrum; we obtain two-sided estimates for them. It is also shown that the eigenvalues emerge from the threshold of the essential spectrum as the window passes through certain critical shapes. We prove the necessary condition for the window to be of critical shape. Under an additional assumption we show that this condition is sufficient and obtain the asymptotic expansion for the emerging eigenvalue and for the associated eigenfunction.

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1. Introduction

There are a number of works studying the spectral properties of the Dirichlet Laplacian in unbounded domains such as infinite planar strips or three-dimensional layers with some perturbations. The interest is stimulated by the applications of such models in quantum mechanics, in particular, in the theory of quantum waveguides. In the case the perturbation is absent, the system is trivial due to the natural separation of variables, while the presence usually leads to various phenomena interesting both from physical and mathematical point of views.

One of the possible systems attracting much attention is two adjacent parallel strips or layers coupled by the window(s) being bounded domain(s) cut out in the common boundary. The two-dimensional case was studied quite intensively, we refer here to [B, ESTV, BE, EV, BEG, BGRS, G3] (see also references therein). It was shown that the perturbation by the window(s) is negative, i.e., it leads to the presence of the isolated bound states below the essential spectrum; the latter is invariant w.r.t. the window(s). In the one-window case it

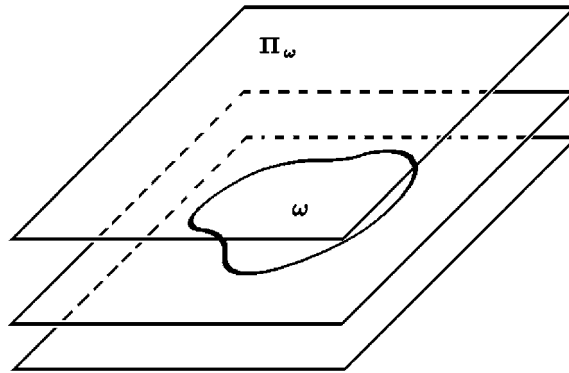


Figure 1. Window-coupled layers.

was shown in [B, ESTV, BEG] that by widening the window one can produce any prescribed number of isolated eigenvalues. They appear when the window's length passes through certain critical values; this phenomenon was studied in details and the asymptotics expansions for the emerging eigenvalues were obtained, see [B, BEG, G3].

In the three-dimensional case corresponding to window-coupled layers Exner and Vugalter showed that a small window generates one simple isolated eigenvalue emerging from the threshold of the essential spectrum [EV]. They also obtained two-sided asymptotic estimates for the eigenvalue. The asymptotics expansion for this eigenvalue has been constructed formally in [P]. In the present paper, we treat the same system but for a finite window. The presence of a window leads to a non-empty discrete spectrum; we obtain two-sided estimates for the eigenvalues. We show that the enlargement of the window produces new isolated eigenvalues emerging from the continuum, and it happens in the way similar to the two-dimensional case. Namely, there are critical shapes of the window so that enlarging the latter one generates a new eigenvalue below the threshold no matter how the increment is small. We show that the necessary condition for such eigenvalue to emerge is the presence of a non-trivial bounded resonance solution corresponding to the threshold of the essential spectrum. We describe all possible resonance solutions. We also prove that the presence of the bounded non-trivial resonance solution of certain type is sufficient to generate an eigenvalue below the essential spectrum. We also give the leading terms of the asymptotics expansions for this eigenvalue and for the associated eigenfunction.

2. Formulation of the problem and the main results

Let $x' = (x_1, x_2)$, $x = (x', x_3)$ be Cartesian coordinates in \mathbb{R}^2 and \mathbb{R}^3 , respectively, and $\omega \subset \mathbb{R}^2$ be a bounded simply connected domain having infinitely differentiable smooth boundary. We denote $\Pi_\omega := \{x : x_3 \in (-d, 0) \cup (0, \pi)\} \cup \omega$, $d \leq \pi$. In what follows the set $\omega \times \{0\}$ is referred to as window (cf figure 1).

The main object of our study is the Dirichlet Laplacian in Π_ω introduced rigorously as associated with the sesquilinear form

$$\mathfrak{h}_\omega[u, v] := (\nabla u, \nabla v)_{L_2(\Pi_\omega)}$$

on $\dot{W}_2^1(\Pi_\omega)$, and we indicate such operator as \mathcal{H}_ω . Hereinafter for a domain $S \subseteq \Pi_\omega$ by $\dot{W}_2^j(S)$ we denote the subset of the functions in $W_2^j(S)$ vanishing on ∂S . Our main aim is to study the spectrum of \mathcal{H}_ω .

In order to present the main results we require additional notations. Assuming $\omega \neq \emptyset$, in a small neighbourhood of $\partial\omega$ we introduce coordinates (τ, s) , where s is the arc length of $\partial\omega$ and τ is the distance to a point measured in the direction of the outward normal to $\partial\omega$. By (r, θ) we denote the polar coordinates corresponding to (τ, x_3) .

Let $\chi = \chi(t) \in C^\infty(\mathbb{R})$ be a cut-off function vanishing as $t > 1/3$ and equalling one as $t < 1/4$. Given $\delta > 0$ small enough, by \mathfrak{Y}_δ we denote the set of the functions

$$u(x) = \left(a_0 + \sum_{j=1}^{\infty} e^{-\frac{2\pi jr}{s_0}} \left(\frac{a_j}{\sqrt{j}} \cos \frac{2\pi js}{s_0} + \frac{\tilde{a}_j}{\sqrt{j}} \sin \frac{2\pi js}{s_0} \right) \right) \sqrt{r} \sin \frac{\theta}{2},$$

$$a_j, \tilde{a}_j \in \mathbb{C}, \quad \|u\|_{\mathfrak{Y}_\delta}^2 := |a_0|^2 + \sum_{j=1}^{\infty} (|a_j|^2 + |\tilde{a}_j|^2) < \infty,$$

defined on $T_\delta := \{x : r < \delta\}$. Here s_0 is the length of $\partial\omega$. We will show that these functions are well defined (see theorem 2.1).

Given $S \subseteq \Pi_\omega$ and small $\delta > 0$, by $\mathfrak{W}(\delta, S)$ we denote the set of the functions

$$u(x) = u^{(0)}(x) \chi\left(\frac{r}{\delta}\right) + u^{(1)}(x), \tag{2.1}$$

where $u^{(0)} \in \mathfrak{Y}_\delta, u^{(1)} \in W_2^2(S), u = 0$ on $\partial\Pi_\omega \cap S$. We will employ the symbol $\mathcal{D}(\cdot)$ to indicate the domain of an operator.

Theorem 2.1. *Suppose ω is non-empty. Then there exists $\delta_0 > 0$ such that $\mathcal{D}(\mathcal{H}_\omega) = \mathfrak{W}(\delta, \Pi_\omega)$, and for each $u \in \mathcal{D}(\mathcal{H}_\omega)$*

$$\mathcal{H}_\omega u = -2\nabla u^{(0)} \cdot \nabla \chi - u^{(0)} \Delta \chi - \Delta u^{(1)}, \quad \chi = \chi\left(\frac{r}{\delta_0}\right) \tag{2.2}$$

The estimates

$$C_1 \|\mathcal{H}_\omega u\|_{L_2(\Pi_\omega)} \leq \|u^{(0)}\|_{\mathfrak{Y}_\delta} + \|u^{(1)}\|_{W_2^2(\Pi_\omega)} \leq C_2 \|\mathcal{H}_\omega u\|_{L_2(\Pi_\omega)},$$

$$\|u^{(0)}\|_{W_2^1(T_\delta)} + \left\| \frac{\partial}{\partial s} \nabla u^{(0)} \right\|_{L_2(T_\delta)} + \left\| r \frac{\partial^2 u^{(0)}}{\partial \tau^2} \right\|_{L_2(T_\delta)} \tag{2.3}$$

$$+ \left\| r \frac{\partial^2 u^{(0)}}{\partial \tau \partial x_3} \right\|_{L_2(T_\delta)} + \left\| r \frac{\partial^2 u^{(0)}}{\partial x_3^2} \right\|_{L_2(T_\delta)} \leq C \|u^{(0)}\|_{\mathfrak{Y}_\delta}.$$

hold true, where the constants $C, C_i > 0$ are independent of $u^{(0)}$ and $u^{(1)}$.

Let $\lambda_i = \lambda_i(\omega)$ be the isolated eigenvalues of \mathcal{H}_ω taken counting multiplicity and ordered in the non-decreasing order. By $\sigma(\cdot), \sigma_{\text{ess}}(\cdot), \sigma_{\text{disc}}(\cdot)$ we denote the spectrum, the essential spectrum and the discrete one of an operator, respectively. We will also use the symbol $\#A$ to indicate the number of the elements in a set A .

Theorem 2.2. *The essential spectrum of \mathcal{H}_ω coincides with $[1, +\infty)$. The discrete spectrum consists of a finite number of the eigenvalues satisfying inequalities*

$$\frac{\pi^2}{(\pi + d)^2} + \mu_i^{(N)} \leq \lambda_i(\omega) \leq \frac{\pi^2}{(\pi + d)^2} + \mu_i^{(D)}, \tag{2.4}$$

where $\mu_i^{(N)}$ and $\mu_i^{(D)}$ are the eigenvalues of the Neumann and Dirichlet Laplacian in ω , respectively. The number of the eigenvalues of \mathcal{H}_ω is estimated as

$$\# \left\{ \mu_i^{(D)} : \mu_i^{(D)} < \frac{2\pi d + d^2}{(\pi + d)^2} \right\} \leq \# \sigma_{\text{disc}}(\mathcal{H}_\omega) \leq \# \left\{ \mu_i^{(N)} : \mu_i^{(N)} < \frac{2\pi d + d^2}{(\pi + d)^2} \right\}. \tag{2.5}$$

We denote $B_\rho(c) := \{x' : |x' - c| < \rho\}$.

Theorem 2.3. *Let $\omega = \omega(t) \subset \mathbb{R}^2$ be a family of bounded simply connected domains having infinitely differentiable boundary and satisfying the following assumption:*

(A1) *For each $t_0 \in (0, +\infty)$ and t close to t_0 there exist diffeomorphism $\mathcal{M}(t_0, t) \in C^3$ defined in the vicinity of $\omega(t_0)$ such that $\mathcal{M}(t_0, t)\omega(t_0) = \omega(t)$, $\mathcal{M}_0(t_0, t_0) = I$; the components of $\mathcal{M}(t_0, t)$ and their derivatives up to the third order are jointly continuous w.r.t. spatial variables and t .*

Then the eigenvalues of $\mathcal{H}_{\omega(t)}$ are continuous w.r.t. t . If, in addition,

(A2) *There exist $\rho_i = \rho_i(t)$, $i = 1, 2$, such that $B_{\rho_1(t)}(0) \subset \omega(t) \subset B_{\rho_2(t)}(0)$, $t \in [0, +\infty)$ and*

$$\lim_{t \rightarrow +\infty} \rho_1(t) = +\infty, \quad \lim_{t \rightarrow +0} \rho_2(t) = 0; \quad (2.6)$$

$\omega(t_1) \subset \omega(t_2)$ for all $t_1 < t_2$; there exists an infinite sequence $0 = t_1 < t_2 \leq t_3 \leq \dots$, such that $\#\sigma_{\text{disc}}(\mathcal{H}_{\omega(t)}) = n$, $t \in (t_n, t_{n+1}]$, $t_n \rightarrow +\infty$, $n \rightarrow +\infty$ and $\lambda_n(\omega(t)) \rightarrow 1 - 0$, $t \rightarrow t_n + 0$.

This theorem states that there exist critical shapes of ω such that by enlarging them one produces new eigenvalue(s) below the essential spectrum. The next part of our results describes how such eigenvalues emerge. First we state

Lemma 2.4. *The problem*

$$-\Delta \Psi = \Psi \quad \text{in } \Pi_\omega, \quad \Psi = 0 \quad \text{on } \partial \Pi_\omega \quad (2.7)$$

has at most finite number of bounded non-trivial solutions assumed to be even w.r.t. x_3 if $d = \pi$. They can be chosen so that there is at most one solution behaving at infinity as

$$\Psi = \sin x_3 + \mathcal{O}(|x'|^{-2}), \quad x' \rightarrow +\infty, \quad x_3 \in (0, \pi); \quad (2.8)$$

at most two solutions behaving as

$$\Psi = \frac{c_1 x_1 + c_2 x_2}{|x'|^2} \sin x_3 + \mathcal{O}(|x'|^{-3}), \quad x' \rightarrow +\infty, \quad x_3 \in (0, \pi), \quad |c_1|^2 + |c_2|^2 = 1; \quad (2.9)$$

and a finite number of solutions belonging to $L_2(\Pi_\omega)$. Each of these solutions is infinitely differentiable up to the boundary except $\partial \omega \times \{0\}$, while in the vicinity of $\partial \omega \times \{0\}$ it behaves as

$$\Psi(x) = l_\Psi(s) \sqrt{r} \sin \frac{\theta}{2} + \mathcal{O}(r), \quad r \rightarrow 0, \quad (2.10)$$

where $l_\Psi \in C^\infty(\partial \omega)$.

Given ω , we consider the family of bounded domains $\omega_\varepsilon \subset \mathbb{R}^2$ whose boundaries are $\partial \omega_\varepsilon := \{x' : \tau = \varepsilon \beta(s)\}$, where $\varepsilon \rightarrow +0$ and $\beta \in C^\infty(\partial \omega)$ is an arbitrary function.

Theorem 2.5. *Suppose the problem (2.7) has no bounded non-trivial solution assumed to be even w.r.t. x_3 if $d = \pi$. Then the operator $\mathcal{H}_{\omega_\varepsilon}$ has no eigenvalues converging to one as $\varepsilon \rightarrow +0$.*

We introduce two-valued symbol, $\gamma := 1$, if $d < \pi$, and $\gamma := 2$, if $d = \pi$.

Theorem 2.6. *Suppose the problem (2.7) has the unique bounded solution Ψ assumed to be even w.r.t. x_3 if $d = \pi$ and this solution satisfies (2.8). Then $l_\Psi \neq 0$, and there exists the unique solution $\tilde{\Psi}$ to (2.7) satisfying the conditions*

$$\tilde{\Psi}(x) = \frac{l_\Psi(s)\beta(s)}{2\sqrt{r}} \sin \frac{\theta}{2} + l_{\tilde{\Psi}}(s)\sqrt{r} \sin \frac{\theta}{2} + \mathcal{O}(r), \quad r \rightarrow 0, \tag{2.11}$$

$$\tilde{\Psi}(x) = \tilde{c} \ln |x'| \sin x_3 + \mathcal{O}(|x'|^{-1}), \quad |x'| \rightarrow +\infty, \quad x_3 \in (0, \pi),$$

where $\tilde{l}_\Psi \in C^\infty(\partial\omega)$, \tilde{c} is a constant. If

$$i_1 := \frac{1}{2\gamma} \int_{\partial\omega} \beta l_\Psi^2 ds > 0; \quad \text{or} \quad i_1 = 0, \quad i_2 := \frac{1}{2\gamma} \int_{\partial\omega} \beta l_\Psi l_{\tilde{\Psi}} ds > 0, \tag{2.12}$$

there exists the unique eigenvalue of $\mathcal{H}_{\omega_\varepsilon}$ converging to $1 - 0$ as $\varepsilon \rightarrow +0$. It is simple, and

$$\lambda_\varepsilon = 1 - 4 \exp\left(-2\mathbf{C} + 2\frac{i_2}{i_1}\right) \exp\left(-\frac{2}{\varepsilon i_1}\right) (1 + \mathcal{O}(\varepsilon)), \quad \text{if } i_1 > 0, \tag{2.13}$$

$$\lambda_\varepsilon = 1 - \exp\left(-\frac{2}{\varepsilon^2 i_2}\right) (c + \mathcal{O}(\varepsilon)), \quad \text{if } i_1 = 0, \quad i_2 > 0,$$

where c is a constant and \mathbf{C} is the Euler constant. The associated eigenfunction satisfies the identity

$$\psi_\varepsilon = \Psi + \mathcal{O}(\sqrt{\varepsilon}) \tag{2.14}$$

in the norms of $W_2^1(S)$ and $W_2^2(S \setminus T_\delta)$ for each bounded fixed domain $S \subset \Pi_{\omega_\varepsilon}$ and $\delta > 0$. It decays exponentially at infinity,

$$\psi_\varepsilon = \mathcal{O}(\varepsilon^{-\sqrt{1-\lambda_\varepsilon}} |x'|^{-1}), \quad |x'| \rightarrow +\infty.$$

If

$$i_1 < 0; \quad \text{or} \quad i_1 = 0, \quad i_2 < 0, \tag{2.15}$$

the operator $\mathcal{H}_{\omega_\varepsilon}$ has no eigenvalues converging to $1 - 0$ as $\varepsilon \rightarrow +0$.

We observe that the leading terms in the asymptotics (2.13) are discontinuous as $d \rightarrow \pi$; this is due to the presence of γ in the formulae. The similar phenomenon was found formally in [P] in the case of small window. We note that it occurs in the two-dimensional case too, see [B].

Theorem 2.5 states that the necessary condition for the eigenvalues to emerge is the presence of a bounded non-trivial solution to (2.7). There are a number of cases corresponding to various non-trivial solutions. One of the possible cases treats theorem 2.6 and other cases remain open. It is an interesting question to obtain the results similar to theorem 2.6 for the remaining cases. In particular, we conjecture that the total multiplicity of the emerging eigenvalues coincides with the number of bounded non-trivial solutions to (2.7). Another conjecture is that if there exists the unique bounded non-trivial solution to (2.7), the eigenvalue emerges if $i_1 > 0$ and does not if $i_1 < 0$. Moreover, if the eigenvalue emerges, its asymptotics should depend on the behaviour at infinity of the non-trivial solution. Namely, we conjecture that

$$\lambda_\varepsilon = 1 - c \frac{\varepsilon}{|\ln \varepsilon|} + \dots, \tag{2.16}$$

if the non-trivial solution satisfies (2.9), and

$$\lambda_\varepsilon = 1 - c\varepsilon + \dots, \tag{2.17}$$

if the non-trivial solution belongs to $L_2(\Pi)$, where c are some constants. One of the motivations to these asymptotics is the results of [KS] where the two-dimensional Schrödinger operator on the plane perturbed by a fast decaying potential was considered. They addressed the same question on describing the behaviour of the eigenvalues emerging from the threshold of the essential spectrum. The asymptotics similar to (2.13) were shown to occur in some cases, while in the other cases the asymptotics similar to (2.16), (2.17) were valid.

3. Domain of \mathcal{H}_ω

The section is devoted to the proof of theorem 2.1. We begin with the series of auxiliary lemmas and notations. We denote $\Omega_\delta := \{(\tau, x_3) : r < \delta\} \setminus \{(\tau, x_3) : x_3 = 0, \tau > 0\}$, where $\delta > 0$ is small enough.

Lemma 3.1. *For each $g \in L_2(\Omega_\delta)$ there exists the unique generalized solution $v \in \dot{W}_2^1(\Omega)$ to the problem*

$$\Delta_{\tau,x_3} v = g \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega. \tag{3.1}$$

It can be represented as $v = v^{(0)} + v^{(1)}$, $v^{(0)} = \alpha\sqrt{r} \sin \frac{\theta}{2}$, where $v^{(1)} \in \dot{W}_2^2(\Omega)$. The estimate

$$|\alpha| + \|v^{(1)}\|_{W_2^2(\Omega_\delta)} \leq C \|g\|_{L_2(\Omega_\delta)} \tag{3.2}$$

holds true, where the constant C is independent of g and δ .

Proof. It is sufficient to give the proof for two subcases corresponding to the function g being odd or even w.r.t. x_3 . In both cases, the unique solvability of (3.1) follows from the standard results in the theory of generalized solutions to elliptic boundary value problems.

If g is odd, the generalized solution to (3.1) is odd w.r.t. x_3 and hence $v = 0$ as $x_3 = 0$. Thus, this function solves the boundary value problem like (3.1) but in the half-disc $\Omega_\delta \cap \{(\tau, x_3) : x_3 > 0\}$. By the smoothness improving theorems we thus obtain that $v \in \dot{W}_2^2(\Omega_\delta)$, $\alpha = 0$, and the estimate (3.2) is valid.

Suppose now that g is even w.r.t. x_3 . We expand g into the Fourier series

$$g(\tau, x_3) = \sum_{j=0}^{\infty} g_{2j+1}(r) \sin \frac{2j+1}{2}\theta, \quad g_p(r) := \frac{1}{\pi} \int_0^{2\pi} g(\tau, x_3) \sin \frac{p\theta}{2} d\theta,$$

which holds true in $L_2(\Omega_\delta)$ -norm. This fact can be established by analogy with the proof of lemma 3.2 in [B]. The Parseval identity

$$\|g\|_{L_2(\Omega)}^2 = \pi \sum_{j=0}^{\infty} \int_0^\delta |g_{2j+1}(r)|^2 r dr \tag{3.3}$$

is valid. We now solve (3.1) by the separation of variables,

$$v(\tau, x_3) = \sum_{j=0}^{\infty} v_{2j+1}(r) \sin \frac{2j+1}{2}\theta, \tag{3.4}$$

$$v_p(r) := \frac{r^{\frac{p}{2}}}{p} \int_\delta^r t^{-\frac{p}{2}+1} g_p(t) dt - \frac{r^{-\frac{p}{2}}}{p} \int_0^r t^{\frac{p}{2}+1} g_p(t) dt + \frac{r^{\frac{p}{2}} \delta^{-p}}{p} \int_0^\delta t^{\frac{p}{2}+1} g_p(t) dt.$$

Let us first analyse the first term in this series. We define

$$\tilde{v}_1(r) := v_1(r) - \alpha\sqrt{r} = r^{\frac{1}{2}} \int_0^r t^{\frac{1}{2}} g_1(t) dt - r^{-\frac{1}{2}} \int_0^r t^{\frac{3}{2}} g_1(t) dt,$$

$$\alpha := \int_0^\delta r^{-\frac{1}{2}} \left(\frac{r}{\delta} - 1\right) g_1 dr = \frac{1}{\pi} \int_{\Omega_\delta} r^{-\frac{1}{2}} \left(\frac{r}{\delta} - 1\right) g \sin \frac{\theta}{2} d\tau dx_3.$$

It is easy to estimate the constant α :

$$|\alpha|^2 \leq \frac{\|g\|_{L_2(\Omega_\delta)}^2}{\pi^2} \int_{\Omega_\delta} r^{-1} \left(\frac{r}{\delta} - 1\right)^2 d\tau dx_3 = \frac{2\delta}{3\pi} \|g\|_{L_2(\Omega_\delta)}^2. \tag{3.5}$$

Employing the estimate

$$\int_0^r t^{\frac{3}{2}} |g_1(t)| dt \leq r \int_0^r t^{\frac{1}{2}} |g_1(t)| dt,$$

and integrating by parts, we check that

$$\begin{aligned} \left\| \tilde{v}_1(r) \sin \frac{\theta}{2} \right\|_{W_2^2(\Omega_\delta)}^2 &\leq C \int_0^\delta (|\tilde{v}_1''|^2 r + |\tilde{v}_1'|^2 r^{-1} + |\tilde{v}_1|^2 r^{-3}) \, dr \\ &\leq C \int_0^\delta r^{-2} \left(\int_0^r t^{\frac{1}{2}} |g_1(t)|^2 \, dt \right)^2 \, dr \\ &\leq C \int_0^\delta r^{-\frac{3}{2}} \int_0^r t^{\frac{3}{2}} |g_1(t)|^2 \, dt \, dr \\ &\leq C \int_0^\delta r |g_1(r)|^2 \, dr \leq C \|g\|_{L_2(\Omega_\delta)}^2, \end{aligned} \tag{3.6}$$

where the constant C is independent of g and δ . In view of the inequality obtained and (3.5) it is sufficient to show that the series $\sum_{j=1}^\infty v_{2j+1}(r) \sin \frac{(2j+1)\theta}{2}$ converges in $W_2^2(\Omega_\delta)$ -norm and to estimate the norm of the sum by $\|g\|_{L_2(\Omega_\delta)}$. More precisely, we should show that

$$\sum_{j=1}^\infty \left\| v_{2j+1}(r) \sin \frac{(2j+1)\theta}{2} \right\|_{W_2^2(\Omega_\delta)} \leq C \|g\|_{L_2(\Omega_\delta)}.$$

Employing the definition of v_p , (3.3), and the estimate

$$\left| \int_0^\delta t^{\frac{p}{2}+1} g_p(t) \, dt \right|^2 \leq \frac{\delta^{p+2}}{p+2} \int_0^\delta r |g_p(r)|^2 \, dr,$$

we see that it is sufficient to check that

$$\sum_{j=1}^\infty j^2 \int_0^\delta \left(r^{2j-2} \left(\int_r^\delta t^{-j+\frac{1}{2}} g_j(t) \, dt \right)^2 + r^{-2j-4} \left(\int_0^r t^{j+\frac{3}{2}} g_j(t) \, dt \right)^2 \right) \, dr \leq C \|g\|_{L_2(\Omega_\delta)}^2,$$

where the constant C is independent of g and δ . This estimate follows from (3.3) and the chain of inequalities

$$\begin{aligned} \int_0^\delta r^{2j-2} \left(\int_r^\delta t^{-j+\frac{1}{2}} g_j(t) \, dt \right)^2 \, dr &\leq \int_0^\delta \frac{2r^{j+\frac{1}{2}}}{5-2j} \left(1 - \frac{r^{j-\frac{5}{2}}}{\delta^{j-\frac{5}{2}}} \right) \int_r^\delta t^{-j-\frac{1}{2}} |g_j(t)|^2 \, dt \, dr \\ &\leq \int_0^\delta \frac{2r^{j+\frac{1}{2}}}{|5-2j|} \int_r^\delta t^{-j-\frac{1}{2}} |g_j(t)|^2 \, dt \, dr \leq \frac{4}{|5-2j|(2j+3)} \int_0^\delta r |g_j(r)|^2 \, dr, \\ \int_0^\delta r^{-2j-4} \left(\int_0^r t^{j+\frac{3}{2}} g_j(t) \, dt \right)^2 \, dr &\leq \int_0^\delta \frac{2r^{-j-\frac{1}{2}}}{2j+7} \int_0^r t^{j+\frac{1}{2}} |g_j(t)|^2 \, dt \, dr \\ &\leq \frac{4}{(2j+7)(2j-1)} \int_0^\delta r |g_j(r)|^2 \, dr, \end{aligned}$$

where we have integrated by parts. □

Lemma 3.2. For each $f \in L_2(T_\delta)$ there exists the unique generalized solution $u \in \dot{W}_2^1(T_\delta)$ to the problem

$$\Delta_{\tau, x_3, s} u = f \quad \text{in } T_\delta, \quad u = 0 \quad \text{on } \partial T_\delta. \tag{3.7}$$

It can be represented as

$$u = u^{(0)} + u^{(1)}, \quad u^{(0)} \in \mathfrak{A}_\delta, \quad u^{(1)} \in W_2^2(T_\delta). \tag{3.8}$$

The estimates (2.3) and

$$\|u^{(0)}\|_{\mathfrak{A}_\delta} + \|u^{(1)}\|_{W_2^2(T_\delta)} \leq C \|f\|_{L_2(T_\delta)} \tag{3.9}$$

are valid.

Proof. The unique solvability of (3.7) is obvious. We separate variables and obtain

$$\begin{aligned} f(x) &= f_0(\tau, x_3) + \sum_{j=1}^{\infty} \left(f_j(\tau, x_3) \cos \frac{2\pi js}{s_0} + \tilde{f}_j(\tau, x_3) \sin \frac{2\pi js}{s_0} \right), \\ f_0 &= \frac{1}{s_0} \int_0^{s_0} f \, ds, \quad f_j = \frac{2}{s_0} \int_0^{s_0} f \cos \frac{2\pi js}{s_0} \, ds, \quad \tilde{f}_j = \frac{2}{s_0} \int_0^{s_0} f \sin \frac{2\pi js}{s_0} \, ds, \\ u(x) &= u_0(\tau, x_3) + \sum_{j=1}^{\infty} \left(u_j(\tau, x_3) \cos \frac{2\pi js}{s_0} + \tilde{u}_j(\tau, x_3) \sin \frac{2\pi js}{s_0} \right), \end{aligned} \quad (3.10)$$

where the series for f converges in $L_2(T_\delta)$, and the coefficients of (3.10) are the generalized solutions to

$$(\Delta_{\tau, x_3} - N^2)v = g \quad \text{in } \Omega_\delta, \quad v = 0 \quad \text{on } \partial\Omega_\delta,$$

where $N = 2\pi j/s_0$ and $g = f_j$ or $g = \tilde{f}_j$. These problems are uniquely solvable in $\dot{W}_2^1(\Omega_\delta)$. By [K, chapter V, section 3.5, equation (3.16)] and the identity

$$\|\nabla v\|_{L_2(\Omega_\delta)}^2 - N^2\|v\|_{L_2(\Omega_\delta)}^2 = (g, v)_{L_2(\Omega_\delta)}$$

we have the estimates

$$\|v\|_{L_2(\Omega_\delta)} \leq \frac{C}{N^2 + 1} \|g\|_{L_2(\Omega_\delta)}, \quad \|v\|_{W_2^1(\Omega_\delta)} \leq \frac{C}{N + 1} \|g\|_{L_2(\Omega_\delta)}, \quad (3.11)$$

where the constant C is independent of g , N and δ . Thus, the series (3.10) converges in $W_2^1(\Omega_\delta)$ -norm and therefore gives the generalized solution to (3.7). This solution also solves (3.1), where the right-hand side of the equation is $(g + N^2v)$. We take into account (3.11) and apply lemma 3.1 to conclude that the function v can be represented as $v = \alpha\sqrt{r} \sin \frac{\theta}{2} + v^{(1)}$, where α and $v^{(1)} \in W_2^2(\Omega_\delta)$ satisfy (3.2).

Let us estimate α more precisely. It follows from lemma 3.1 that the first term in the series (3.4) for v satisfies the relations

$$v_0(r) \sin \frac{\theta}{2} = \alpha\sqrt{r} \sin \frac{\theta}{2} + \tilde{v}_0(\tau, x_3), \quad v_0(r) = \frac{1}{\pi} \int_0^{2\pi} v \sin \frac{\theta}{2} \, d\theta,$$

where $\tilde{v}_0 \in W_2^2(\Omega_\delta)$. The function v_0 solves the problem

$$\left(\frac{d}{dr} r \frac{d}{dr} - \frac{1}{4r^2} - N^2 \right) v_0 = g_0 \quad \text{in } (0, \delta), \quad v_0(\delta) = 0, \quad g_0 := \frac{1}{\pi} \int_0^{2\pi} g \sin \frac{\theta}{2} \, d\theta,$$

and obeys the condition $v_0(r) \sin \frac{\theta}{2} \in \dot{W}_2^1(\Omega_\delta)$. Hence,

$$v_0(r) = \int_0^r g_0(t) \frac{\sqrt{t} \sinh N(r-t)}{N\sqrt{r}} \, dt + \frac{\sinh Nr}{N\sqrt{r} \sinh N\delta} \int_0^\delta g_0(t) \sqrt{t} \sinh N(t-\delta) \, dt.$$

Proceeding as in (3.6), we check that

$$\frac{\sin \frac{\theta}{2}}{N\sqrt{r}} \int_0^r g_0(t) \sqrt{t} \sinh N(r-t) \, dt \in W_2^2(\Omega_\delta),$$

and therefore

$$\begin{aligned} \alpha &= \int_0^\delta \frac{\sqrt{t} \sinh N(t-\delta)}{\sinh N\delta} g_0(t) \, dt, \\ |\alpha|^2 &\leq \int_0^\delta \frac{\sinh^2 N(\delta-t)}{\sinh^2 N\delta} \, dt \int_0^\delta t |g_0(t)|^2 \, dt \leq \frac{C}{N} \|g\|_{L_2(\Omega_\delta)}, \end{aligned} \quad (3.12)$$

where

$$C = \max_{[0,+\infty)} \frac{\sinh 2t - 2t}{4 \sinh^2 t} < \infty.$$

It is easy to check that for $N \neq 0$

$$\left\| \sqrt{r} e^{-Nr} \sin \frac{\theta}{2} \right\|_{L_2(\Omega_\delta)}^2 = \pi \int_0^\delta r^2 e^{-2Nr} dr \leq \frac{\pi}{N^3} \int_0^{+\infty} t^2 e^{-2t} dt = \frac{\pi}{4N^3}. \tag{3.13}$$

In the same way one can make sure

$$\left\| \sqrt{r} e^{-Nr} \sin \frac{\theta}{2} \right\|_{W_2^1(\Omega_\delta)} \leq \frac{C}{N}, \quad \left\| \sqrt{r}(e^{-Nr} - 1) \sin \frac{\theta}{2} \right\|_{W_2^2(\Omega_\delta)} \leq CN,$$

where the constant C is independent of N and δ . By (3.11) and (3.2) we conclude now that

$$\begin{aligned} \left\| v - \alpha \sqrt{r} e^{-Nr} \sin \frac{\theta}{2} \right\|_{L_2(\Omega_\delta)} &\leq \frac{C}{N^2 + 1} \|g\|_{L_2(\Omega_\delta)}, \\ \left\| v - \alpha \sqrt{r} e^{-Nr} \sin \frac{\theta}{2} \right\|_{W_2^1(\Omega_\delta)} &\leq \frac{C}{N + 1} \|g\|_{L_2(\Omega_\delta)}, \\ \left\| v^{(1)} - \alpha \sqrt{r}(e^{-Nr} - 1) \sin \frac{\theta}{2} \right\|_{W_2^2(\Omega_\delta)} &\leq C \|g\|_{L_2(\Omega_\delta)}, \end{aligned}$$

where the constant C is independent of g , N and δ . These estimates and (3.12) applied to the coefficients of the series (3.10) lead us to (3.8), (3.9), if we denote the fractions $\alpha \sqrt{N}$ corresponding to u_j and \tilde{u}_j by a_j and \tilde{a}_j . The inequality (2.3) can be checked by estimating the appropriate norms of $\sqrt{r} e^{-Nr}$ in the same way as in (3.13). \square

Lemma 3.3. *There exists $\delta_0 > 0$ small enough such that for any $f \in L_2(T_{\delta_0})$ the generalized solution to*

$$\Delta_x u = f \quad \text{in } T_{\delta_0}, \quad u = 0 \quad \text{on } \partial T_{\delta_0}, \tag{3.14}$$

satisfies (3.8), (3.9) and (2.3).

Proof. Lemma 3.2 implies that the domain of the operator $\Delta_{\tau, x_3, s}$ in T_δ with Dirichlet boundary conditions is $\{u : u \in \mathfrak{D}_\delta \oplus W_2^2(T_\delta), u|_{\partial T_\delta} = 0\}$; the action of this operator reads $\Delta_{\tau, x_3, s}^{(D)} u = \Delta_{\tau, x_3, s} u$.

The Dirichlet Laplacian on T_δ can be written as

$$\begin{aligned} \Delta_x^{(D)} &= \Delta_{\tau, x_3, s}^{(D)} + \mathcal{L}_2 u + \mathcal{L}_1 u, \\ \mathcal{L}_2 &:= \tau \frac{2k - \tau k^2}{(1 - \tau k)^2} \frac{\partial^2}{\partial s^2}, \quad \mathcal{L}_1 := -\frac{k}{1 - \tau k} \frac{\partial}{\partial \tau} - \frac{\tau k'}{(1 - \tau k)^3} \frac{\partial}{\partial s}, \end{aligned}$$

where $k = k(s) \in C^\infty(\partial\omega)$. The operator \mathcal{L}_2 is $\Delta_{\tau, x_3, s}^{(D)}$ -bounded due to (3.9), (2.3), and the bound is estimated by $C\delta$, C is independent of δ . The operator \mathcal{L}_1 is $\Delta_{\tau, x_3, s}^{(D)}$ -compact. Employing now [K, chapter IV, section 1.1, theorem. 1.1], we conclude that the domain of $\Delta_x^{(D)}$ is the same as that of $\Delta_{\tau, x_3, s}^{(D)}$, if δ is small enough. Therefore, the representation (3.8) is valid. The estimates (3.9), (2.3) for the solution to (3.14) follow from that for the solution to (3.7) and [K, chapter IV, section 1.4, theorem 1.16]. \square

Let u be a function in the domain of \mathcal{H}_ω . By the definition, $u \in \mathring{W}_2^1(\Pi_\omega)$, and it is a generalized solution to

$$-\Delta_x u = f \quad \text{in } \Pi_\omega, \quad u = 0 \quad \text{on } \partial \Pi_\omega, \tag{3.15}$$

where $f = \mathcal{H}_\omega u$. Using the smoothness improving theorems one can show that $u \in W_2^2(S)$ for any $S \in \Pi_\omega \setminus T_\delta$, $\delta > 0$, and hence

$$\mathcal{H}_\omega u = -\Delta_x u. \quad (3.16)$$

It is also clear that

$$\|u\|_{W_2^1(T_\delta)} \leq C \|f\|_{L_2(T_\delta)}. \quad (3.17)$$

We denote

$$\tilde{u}(x) := \left(1 - \chi\left(\frac{r}{2\delta}\right)\right) u(x) \in \mathring{W}_2^2(\Pi_\omega).$$

Employing (3.17) and proceeding as in the proof of lemma 7.1 in [LU, chapter 3, section 7], one can check that

$$\|\tilde{u}\|_{W_2^2(\Pi_\omega)} \leq C \|f\|_{L_2(\Pi_\omega)}.$$

The function $\hat{u}(x) := u(x)\chi\left(\frac{r}{2\delta}\right)$ is the solution to (3.14) with the right-hand side

$$\tilde{f} := -f - 2\nabla_x u \cdot \nabla_x \chi - u \Delta_x \chi, \quad \chi = \chi\left(\frac{r}{2\delta}\right).$$

In view of (3.17) we have

$$\|\tilde{f}\|_{L_2(T_\delta)} \leq C(\delta) \|f\|_{L_2(T_\delta)}.$$

Employing now lemma 3.3, we conclude that the representation (3.8) and the estimates (3.9), (2.3) hold true. It remains to note that by (3.8)

$$\hat{u} = \chi\left(\frac{r}{\delta}\right) \hat{u} = \chi\left(\frac{r}{\delta}\right) \hat{u}^{(0)} + \chi\left(\frac{r}{\delta}\right) \hat{u}^{(1)}.$$

Denoting now $u^{(1)} := \tilde{u} + \chi\left(\frac{r}{\delta}\right) \hat{u}^{(1)}$, we conclude that the representation (2.1) holds true.

If u is given by (2.1), it is easy to check that $u \in \mathring{W}_2^1(\Pi_\omega)$ and it is the generalized solution to the problem (3.15), where the right-hand side is that of (2.2). Thus, u belongs to the domain of \mathcal{H}_ω . To prove (2.2), it is sufficient to substitute (2.1) into (3.16). The proof of theorem 2.1 is complete.

4. Estimates and continuity of the eigenvalues

In this section we prove theorems 2.2 and 2.3.

Proof of theorem 2.2. The main idea of the proof is borrowed from [ESTV, section II]. We introduce additional boundary $\partial\omega \times (-d, \pi)$ and impose in turn Dirichlet and Neumann boundary condition on it. As a result, we have two direct sums $\mathcal{H}_{\text{int}}^{(D)} \oplus \mathcal{H}_{\text{ext}}^{(D)}$ and $\mathcal{H}_{\text{int}}^{(N)} \oplus \mathcal{H}_{\text{ext}}^{(N)}$, where $\mathcal{H}_{\text{int}}^{(D)}$ is the Dirichlet Laplacian in $\omega \times (-d, \pi)$ and $\mathcal{H}_{\text{ext}}^{(D)}$ is the Dirichlet Laplacian in $\Pi_\omega \setminus (\omega \times (-d, \pi))$. The operators $\mathcal{H}_{\text{int}}^{(N)}$, $\mathcal{H}_{\text{ext}}^{(N)}$ are introduced in the same way; the difference is the boundary condition on $\partial\omega \times (-d, \pi)$ which is the Neumann one.

The identities

$$\sigma_{\text{ess}}(\mathcal{H}_\omega) = \sigma(\mathcal{H}_{\text{ext}}^{(D)}) = \sigma_{\text{ess}}(\mathcal{H}_{\text{ext}}^{(D)}) = \sigma(\mathcal{H}_{\text{ext}}^{(N)}) = \sigma_{\text{ess}}(\mathcal{H}_{\text{ext}}^{(N)}) = [1, +\infty)$$

can be proven in the same way as the similar identity in the proof of theorem 2.1 in [BE]. The eigenvalues of $\mathcal{H}_{\text{int}}^{(N)}$, $\mathcal{H}_{\text{int}}^{(D)}$ are calculated by the separation of variables:

$$\lambda_{i,j}(\mathcal{H}_{\text{int}}^{(D)}) = \mu_i^{(D)} + \frac{\pi^2 j^2}{(\pi + d)^2}, \quad \lambda_{i,j}(\mathcal{H}_{\text{int}}^{(N)}) = \mu_i^{(N)} + \frac{\pi^2 j^2}{(\pi + d)^2}, \quad i, j \geq 1.$$

It is clear that $\pi^2 j^2 / (\pi + d)^2 > 1, j \geq 2$. Taking into account this inequality and standard bracketing [RS, chapter XIII, section 15], we arrive at the estimates (2.4). The estimates (2.5) follow from (2.4). \square

Proof of theorem 2.3. Let us first prove that the eigenvalues of $\mathcal{H}_{\omega(t)}$ are continuous w.r.t. t . Given $t_0 \in (0, +\infty)$ and t close to t_0 , we introduce new variables as

$$\begin{aligned} \tilde{x} &= (\tilde{x}', \tilde{x}_3), & \tilde{x}' &= \chi\left(\frac{r}{\delta}\right) \mathcal{M}(t_0, t)x' + \left(1 - \chi\left(\frac{r}{\delta}\right)\right) x', \\ \tilde{x}_3 &= \left(b(x', t)\chi\left(\frac{r}{\delta}\right) + 1 - \chi\left(\frac{r}{\delta}\right)\right) x_3, & b(x', t) &= \sqrt{\left(\frac{\partial \tilde{\tau}}{\partial \tau}\right)^2 + \frac{1}{(1 - \tau k)^2} \left(\frac{\partial \tilde{s}}{\partial s}\right)^2} \end{aligned}$$

where $(\tilde{\tau}, \tilde{s})$ are associated with \tilde{x}' in the same way as (τ, s) with x' . Keeping in mind assumption (A1), one can easily make sure that the variables \tilde{x} are well defined for t sufficiently close to t_0 , and the domain $\Pi_{\omega(t_0)}$ is mapped onto $\Pi_{\omega(t)}$ under such change of variables. In the space $L_2(\Pi_{\omega(t_0)}) = L_2(\Pi_{\omega(t)})$ we define a unitary operator

$$(Q(t)u) = \sqrt{q}u(Q(\cdot)), \quad q := \det \left(\frac{\partial x_i}{\partial \tilde{x}_j} \right)_{i,j=1,\dots,3},$$

where Q is defined as $x = Q(\tilde{x})$. By direct calculations we check that

$$Q(t)\mathcal{H}_{\omega(t)}Q^{-1}(t) = b(Q(\cdot), t)\mathcal{H}_{\omega(t_0)} + \mathcal{L}_3, \tag{4.1}$$

$$\begin{aligned} \mathcal{L}_3 &:= b_{12} \frac{\partial^2}{\partial \tau \partial s} + x_3 b_{13} \frac{\partial^2}{\partial \tau \partial x_3} + b_{22} \frac{\partial^2}{\partial s^2} + b_{23} \frac{\partial^2}{\partial s \partial x_3} + x_3 b_{33} \frac{\partial^2}{\partial x_3^2} \\ &\quad + b_1 \frac{\partial}{\partial \tau} + b_2 \frac{\partial}{\partial s} + b_3 \frac{\partial}{\partial x_3} + b_0, \end{aligned} \tag{4.2}$$

where $b_{i,j} = b_{i,j}(x, t) \in C(T_\delta \times (t_0 - c, t_0 + c)), b_i = b_i(x, t) \in C(T_\delta \times (t_0 - c, t_0 + c))$ and $b_{i,j}(x, 0) = b_i(x, 0) = 0$. The supports of $b_{i,j}, b_i$ lie inside T_δ . It is also follows from (A1) that

$$b(x', t) = 1 + \tilde{b}(x', t), \quad \tilde{b}(x', t) \in C_0(T_\delta \times (t_0 - c, t_0 + c)), \quad \tilde{b}(x', 0) = 0.$$

Theorem 2.1 and, in particular, the estimates (2.3) imply that \mathcal{L}_3 is $\mathcal{H}_{\omega(t_0)}$ -bounded. By (4.1) and the last formula for b we conclude now that the difference $(Q(t)\mathcal{H}_{\omega(t)}Q^{-1}(t) - \mathcal{H}_{\omega(t_0)})$ is a small perturbation bounded relatively w.r.t. $\mathcal{H}_{\omega(t_0)}$, and $Q(t_0)\mathcal{H}_{\omega(t_0)}Q^{-1}(t_0) = \mathcal{H}_{\omega(t_0)}$. Thus, the eigenvalues of $Q(t)\mathcal{H}_{\omega(t)}Q^{-1}(t)$ converges to those of $\mathcal{H}_{\omega(t_0)}$. Therefore, the same is true for the eigenvalues of $\mathcal{H}_{\omega(t)}$.

Assume now that $\omega(t_1) \subset \omega(t_2)$ for all $t_1 < t_2$. These are the standard minimax arguments which show that the eigenvalues $\lambda_i(\omega(t))$ are monotonically decreasing functions of t . Hence, to prove the last statement of the theorem it is sufficient to show that for each eigenvalue $\lambda_i(\omega(t))$ there exists t_i such that $\lambda_i(\omega(t)) \rightarrow 1 - 0$, as $t \rightarrow t_i + 0$. Suppose that this is wrong for an eigenvalue $\lambda_j(\omega(t))$ on a sequence $t^{(m)} \rightarrow +0$. In this case $\lambda_1(\omega(t^{(m)})) \leq \lambda_j(\omega(t^{(m)})) \leq c < 1$. At the same time, by [EV, theorem 3.1] and the second identity in (2.6) we have $\lambda_1(\omega(t)) \rightarrow 1 - 0, t \rightarrow +0$, the contradiction. The sequence of critical values t_i is infinite due to (2.4) and the first identity in (2.6). \square

5. Reduction of the resolvent to a compact operator

In this section, we study the boundary value problem

$$-\Delta u = (1 - k^2)u + f \quad \text{in } \Pi_\omega, \quad u = 0 \quad \text{on } \partial\Pi_\omega, \tag{5.1}$$

where $k \in \mathbb{C}$ ranges in a small neighbourhood of zero, $f \in L_2(\Pi_\omega)$, $\text{supp} f \subseteq \Pi_{\omega,\beta} := \Pi_\omega \cap \{x : |x'| < \beta\}$, $\beta > 0$. We choose β so that $\omega \subset \{x' : |x'| < \beta/4\}$. If $d = \pi$, we assume in addition that g is even w.r.t. x_3 and the same is supposed for u . We should also specify the behaviour at infinity for the solutions of (5.1). If $\text{Re } k > 0$, we take the function $u = (\mathcal{H}_\omega - 1 + k^2)^{-1} f$ as the solution to (5.1). For other values of k we will define the analytic continuation of the operator $(\mathcal{H}_\omega - 1 + k^2)^{-1}$. We will do it by the technique employed in [B, section 3], [BEG, section 3.A]. We will also reduce (5.1) to a Fredholm equation in $L_2(\Pi_{\omega,\beta})$ that will be one of the key ingredient in the proof of theorems 2.5 and 2.6.

Let $g \in L_2(\Pi_{\omega,\beta})$, $\text{supp} g \subseteq \Pi_{\omega,\beta}$ and g is even w.r.t. x_3 , if $d = \pi$. By $v = v(x, k)$ we denote the solution to the problem

$$-\Delta v = (1 - k^2)v + g \quad \text{in } \Pi_\emptyset, \quad v = 0 \quad \text{on } \partial\Pi_\emptyset,$$

given by the formulae

$$\begin{aligned} v(x, k) &= \begin{cases} v^+(x, k), & x_3 \in (0, \pi), \\ v^-(x, k), & x_3 \in (-d, 0), \end{cases} \\ v^+(x, k) &= \sum_{j=1}^\infty v_j^+(x', k) \sin jx_3, & v^-(x, k) &= \sum_{j=1}^\infty v_j^-(x', k) \sin \frac{\pi j}{d} x_3, \\ v_j^+(x', k) &:= \frac{1}{2\pi i} \int_{\mathbb{R}^2 \times (0, \pi)} g(y) \text{H}_0(i|x' - y'| \sqrt{j^2 - 1 + k^2}) \sin y_3 \, dy, \\ v_j^-(x', k) &:= \frac{1}{2id} \int_{\mathbb{R}^2 \times (-d, 0)} g(y) \text{H}_0\left(i|x' - y'| \sqrt{\frac{\pi^2 j^2}{d^2} - 1 + k^2}\right) \sin \frac{\pi j}{d} y_3 \, dy, \end{aligned}$$

where H_0 is the Hankel function and $\sqrt{k^2} = k$, while the other roots are specified by the requirement $\sqrt{1} = 1$. If $k = 0$, we introduce the function v_1^+ as

$$v_1^+(x', 0) = \frac{1}{\pi^2} \int_{\mathbb{R}^2 \times (0, \pi)} g(y) \ln|x' - y'| \sin y_3 \, dy,$$

and

$$v_1^-(x', 0) = \frac{1}{\pi^2} \int_{\mathbb{R}^2 \times (-\pi, 0)} g(y) \ln|x' - y'| \sin y_3 \, dy, \quad \text{if } d = \pi.$$

The function v is well defined and belongs to $W_2^2(\Pi_{\emptyset, \tilde{\beta}})$ for all $\tilde{\beta} > 0$ and all considered values of k . This fact can be shown by analogy with the proof of lemma 3.1 in [B].

Consider the problem

$$\Delta w = \Delta v \quad \text{in } \Pi_{\omega,\beta}, \quad w = v \quad \text{on } \partial\Pi_{\omega,\beta}. \tag{5.2}$$

This problem is uniquely solvable in $W_2^1(\Pi_{\omega,\beta})$. We construct the solution to (5.1) as

$$u(x, k) = (\mathcal{A}_1(k)g)(x, k) := w(x, k)\chi(|x'|/\beta) + (1 - \chi(|x'|/\beta))v(x, k). \tag{5.3}$$

This function satisfies the boundary condition in (5.1). Substituting it into equation (5.1), we obtain

$$g + \mathcal{A}_2(k)g = f, \tag{5.4}$$

$$\mathcal{A}_2(k)g := (v - w)(\Delta + 1 - k^2)\chi\left(\frac{|x'|}{\beta}\right) + 2\nabla\chi\left(\frac{|x'|}{\beta}\right) \cdot \nabla(v - w). \tag{5.5}$$

By \mathfrak{A} we denote the set of the operators $\mathcal{A} = \mathcal{A}(k)$ bounded as operators from $L_2(\Pi_{\omega,\beta})$ into $W_2^1(\Pi_{\omega, \tilde{\beta}})$, $\mathfrak{W}(\delta, \tilde{\beta})$ and $W_2^2(\Pi_{\omega, \tilde{\beta}} \setminus T_\delta)$ for each $\tilde{\beta} > 0$, $\delta > 0$ and small k , and such that

the function $\mathcal{A}(k)f$ is real valued for real-valued f and small non-negative k . If an operator $\mathcal{A}(k)$ belongs to \mathfrak{A} and is continuous (uniformly bounded, holomorphic) w.r.t. k as an operator from $L_2(\Pi_\beta)$ into each of aforementioned spaces, we will say shortly that the operator $\mathcal{A}(k)$ belongs to \mathfrak{A} and is continuous (uniformly bounded, holomorphic) w.r.t. k .

We denote

$$a(g) := \frac{1}{\pi^2} \int_{\Pi_{\omega,\beta} \cap \{x: x_3 > 0\}} g(x) \sin x_3 \, dx_3.$$

Repeating the arguments of the proofs of lemmas 3.1 and 3.3–3.5 in [B] and of propositions 3.1 and 3.2 in [BEG], and employing lemma 3.3 one can prove

Lemma 5.1. *Let $k \in \mathbb{C}$ be small enough. The operator $\mathcal{A}_1(k) \in \mathfrak{A}$ is bounded uniformly w.r.t. small k . The operator $\mathcal{A}_2(k)$ is a linear compact operator in $L_2(\Pi_{\omega,\beta})$. For $k \neq 0$ these operators can be represented as*

$$\mathcal{A}_1(k) = \mathcal{A}_3(k^2) + \mathcal{A}_4(k^2) \ln k, \quad \mathcal{A}_2(k) = \mathcal{A}_5(k^2) + \mathcal{A}_6(k^2) \ln k, \quad (5.6)$$

where $\mathcal{A}_3(\cdot), \mathcal{A}_4(\cdot) \in \mathfrak{A}$ are holomorphic, and $\mathcal{A}_5(\cdot), \mathcal{A}_6(\cdot)$ are linear compact operators in $L_2(\Pi_{\omega,\beta})$ being holomorphic w.r.t. k . The functions $\mathcal{A}_i(k^2)f, i = 5, 6$, are real valued if f is real valued and k^2 is small and non-negative. For each $f \in L_2(\Pi_{\omega,\beta})$ there exists a solution to (5.1) given by $u = \mathcal{A}_1(k)g$. This solution behaves at infinity as

$$u(x, k) = c \left(k, \frac{x'}{|x'|} \right) e^{-k|x'|} |x'|^{-\frac{1}{2}} \sin x_3 + \mathcal{O}(e^{-k|x'|} |x'|^{-\frac{3}{2}}), \quad x_3 \in (0, \pi),$$

$$c(k, \xi) = -\frac{\sqrt{2\pi}}{4\sqrt{k}} a((1 + k\xi \cdot x')g(x)) + \mathcal{O}(k^{\frac{3}{2}}), \quad k \rightarrow 0, \quad (5.7)$$

$$u(x, k) = \mathcal{O}(e^{-\sqrt{\frac{\pi^2}{d^2} - 1 + k^2}|x'|} |x'|^{-\frac{3}{2}}), \quad x_3 \in (-d, 0), \quad \text{if } d < \pi,$$

if $k \neq 0$, and

$$u(x, 0) = \left(a(g) \ln|x'| + \frac{c_1 x_1 + c_2 x_2}{|x'|^2} \right) \sin x_3 + \mathcal{O}(|x'|^{-2}), \quad x_3 \in (0, \pi), \quad (5.8)$$

$$u(x, 0) = \mathcal{O}(e^{-\sqrt{\frac{\pi^2}{d^2} - 1}|x'|} |x'|^{-\frac{3}{2}}), \quad x_3 \in (-d, 0), \quad \text{if } d < \pi.$$

For each solution to (5.1) behaving at infinity in accordance with (5.7), (5.8) there exists the unique solution to (5.4) such that $u = \mathcal{A}_1(k)g$.

We denote

$$V_0(x) := \begin{cases} \sin x_3, & x_3 \in (0, \pi), \\ (1 - \gamma) \sin x_3, & x_3 \in (-d, 0). \end{cases}$$

By W_0 we indicate the solution to (5.2) as $v = V_0$. We introduce the functions U_0 by (5.3) via V_0 and W_0 . The next lemma is checked by direct calculations.

Lemma 5.2. *The identities*

$$\begin{aligned} \mathcal{A}_3(0)g &= (C - \ln 2)\mathcal{A}_4(0)g + \mathcal{A}_1(0)g, & \mathcal{A}_4(0)g &= a(g)U_0, \\ \mathcal{A}_5(0)g &= (C - \ln 2)\mathcal{A}_6(0)g + \mathcal{A}_2(0)g, & \mathcal{A}_6(0)g &= -a(g)(\Delta + 1)U_0, \end{aligned}$$

hold true.

Lemma 5.3. *Let $k = 0$. There is a finite number of linear independent non-trivial solutions to (5.1), (5.8) assumed to be even w.r.t. x_3 if $d = \pi$. They can be chosen so that there is at most one solution behaving at infinity*

$$\Psi(x) = \ln|x'| \sin x_3 + \mathcal{O}(|x'|^{-1}), \quad |x'| \rightarrow +\infty, \quad x_3 \in (0, \pi); \quad (5.9)$$

at most two solutions satisfying (2.9), and a finite number of solutions belonging to $L_2(\Pi_\omega)$. Each of these solutions is infinitely differentiable up to the boundary except $\partial\omega \times \{0\}$, and in the vicinity of $\partial\omega \times \{0\}$ it satisfies (2.10).

Proof. The statement on the existence and the number of the solutions follows immediately from (5.8). The claimed smoothness is due to the standard smoothness improving theorems. Formula (2.10) can be checked by analogy with the proof of lemma 4.2 in [B]. \square

Lemma 5.4. *Let $k = 0$. Equation (5.4) is solvable if and only if $(f, \Psi_i)_{L_2(\Pi_{\omega,\beta})} = 0$, where Ψ_i are non-trivial solutions to (5.1), (5.8). If the solvability conditions holds true, there exists the unique solution of (5.4) orthogonal to ϕ in $L_2(\Pi_{\omega,\beta})$.*

Proof. By lemma 5.1 the operator $\mathcal{A}_2(0)$ is compact. Thus, equation (5.4) is solvable if and only if $(f, \phi_i^*)_{L_2(\Pi_{\omega,\beta})} = 0$, where ϕ_i^* are non-trivial solutions to the adjoint equation $\phi_i^* + \mathcal{A}_2^*(0)\phi_i^* = 0$. It is sufficient to show that $\phi_i^* = \Psi_i$. Since the numbers of ϕ_i^* and Ψ_i are same, in view of lemma 5.1 it is sufficient to check that

$$0 = (\Psi_i + \mathcal{A}_2^*(0)\Psi_i, h)_{L_2(\Pi_{\omega,\beta})} = (\Psi_i, h + \mathcal{A}_2(0)h)_{L_2(\Pi_{\omega,\beta})}$$

for all $h \in L_2(\Pi_{\omega,\beta})$. We denote $u := \mathcal{A}_1(0)h$; by the definition of $\mathcal{A}_1(0)$ this function satisfies (5.8). The same formula is valid for Ψ_i . Moreover, $h + \mathcal{A}_2(0)h = -(\Delta + 1)u$. Taking these facts into account and integrating by parts, we obtain

$$(\Psi_i, h + \mathcal{A}_2(0)h)_{L_2(\Pi_{\omega,\beta})} = - \int_{\Pi_{\omega,\beta}} \Psi_i(\Delta + 1)u \, dx = - \int_{\Pi_\omega} u(\Delta + 1)\Psi_i \, dx = 0. \quad \square$$

Proof of lemma 2.4. The most part of the lemma follows from lemma 5.3; we just need to check the statement on the solution satisfying (2.8). If there exists the non-trivial solution u behaving at infinity in accordance with (5.9), the problem (2.7) cannot have a solution Ψ satisfying (2.8). This fact can be proven by integrating by parts in the integral $0 = \int_{\Pi_{\omega,\beta}} u(\Delta + 1)\Psi \, dx$.

Assume that there is no non-trivial solution obeying (5.9); let us prove that in this case there is the unique non-trivial solution behaving at infinity as

$$\Psi(x) = (c \ln|x'| + 1) \sin x_3 + \mathcal{O}(|x'|^{-1}), \quad |x'| \rightarrow +\infty, \quad x_3 \in (0, \pi), \quad c \neq 0. \quad (5.10)$$

We construct it as

$$\begin{aligned} \Psi(x) &= \tilde{\Psi}(x) + \left(1 - \chi\left(\frac{2|x'|}{3\beta}\right)\right) \ln|x'| \sin x_3, & x_3 \in (0, \pi), \\ \Psi(x) &= \tilde{\Psi}(x), & x_3 \in (-d, 0), & \text{if } d < \pi, \\ \Psi(x) &= \tilde{\Psi}(x) + \left(1 - \chi\left(\frac{2|x'|}{3\beta}\right)\right) \ln|x'| \sin x_3, & x_3 \in (-\pi, 0), & \text{if } d = \pi. \end{aligned}$$

It leads us to the problem (5.1) for $\tilde{\Psi}$ with $k = 0$ where

$$\begin{aligned} f(x) &= (\Delta + 1) \left(1 - \chi\left(\frac{2|x'|}{3\beta}\right)\right) \sin x_3, & x_3 \in (0, \pi), \\ f(x) &= (\gamma - 1)(\Delta + 1) \left(1 - \chi\left(\frac{2|x'|}{3\beta}\right)\right) \sin x_3, & x_3 \in (-\pi, 0). \end{aligned}$$

Taking into account lemma 5.1 and integrating by parts, it is easy to check that $(f, \Psi_i)_{L_2(\Pi_{\omega, \beta})} = 0$ for each non-trivial solution Ψ_i of (5.1), (5.8). By lemma 5.4 equation (5.4) is thus solvable that by lemma 5.1 proves the solvability of the problem for $\tilde{\Psi}$. The uniqueness of Ψ follows from (5.10). \square

6. Singularity of the resolvent

In this section, we study the behaviour of the operator $(I + \mathcal{A}_2(k))^{-1}$ in the vicinity of $k = 0$. We recall that, if $d = \pi$, we restrict all the operators on the even w.r.t. x_3 functions. We consider two cases corresponding to different possibilities of the presence of non-trivial solution described in lemmas 2.4 and 5.3. The results of the section will be employed in the proofs of theorems 2.5 and 2.6.

6.1. Absence of decaying and logarithmically growing non-trivial solution

Here we deal with the case when the problem (5.1), (5.8) has no non-trivial solutions described in lemma 5.3. As it has been shown in the proof of lemma 2.4, in this case the problem (2.7) has the unique non-trivial solution satisfying (5.10).

We substitute (5.6) into (5.4) and take into account lemma 5.2. It leads us to

$$g + \mathcal{A}_2(0)g - (\ln k - \ln 2 + C)a(g)(\Delta + 1)U_0 + k^2 \ln k \mathcal{A}_7(k)g = f, \tag{6.1}$$

where

$$\mathcal{A}_7(k) := \frac{\mathcal{A}_5(k^2) - \mathcal{A}_5(0) + (\mathcal{A}_6(k^2) - \mathcal{A}_6(0)) \ln k}{k^2 \ln k}$$

is a compact operator in $L_2(\Pi_{\omega, \beta})$ continuous w.r.t. small real k . The same is true for $(k\mathcal{A}_7(k))'$. Hereinafter the expressions like $\ln k \mathcal{A}_7(k)$ are understood as $(\ln k)\mathcal{A}_7(k)$.

By the assumption the operator $(I + \mathcal{A}_2(0))$ is invertible, and the same is thus true for $(I + \mathcal{A}_2(0) + k^2 \ln k \mathcal{A}_7(k))$. We denote the inverse to the latter as $\mathcal{A}_8(k)$ and apply it to (6.1),

$$g - (\ln k - \ln 2 + C)a(g)\mathcal{A}_8(k)(\Delta + 1)U_0 = \mathcal{A}_8(k)f.$$

Now we apply the functional a to the equation obtained that solves (6.1),

$$a(g) = \frac{a(\mathcal{A}_8(k)f)}{1 - (\ln k - \ln 2 + C)a(\mathcal{A}_8(k)(\Delta + 1)U_0)}, \tag{6.2}$$

$$(I + \mathcal{A}_2(k))^{-1}f = g = \frac{(\ln k - \ln 2 + C)a(\mathcal{A}_8(k)f)\mathcal{A}_8(k)(\Delta + 1)U_0}{1 - (\ln k - \ln 2 + C)a(\mathcal{A}_8(k)(\Delta + 1)U_0)} + \mathcal{A}_8(k)f.$$

Let us prove that the denominator is non-zero. In order to do it, we need

Lemma 6.1. *The identities*

$$\mathcal{A}_1(0)\mathcal{A}_8(0)(\Delta + 1)U_0 + U_0 = \Psi, \quad a((\Delta + 1)U_0) = c,$$

hold true, where Ψ is the unique solution to (2.7), (2.8) and c is from (5.10).

Proof. It is sufficient to prove the former formula, since it implies the latter due to (5.8).

The definition of \mathcal{A}_8 yields that $\mathcal{A}_8(0)(\Delta + 1)U_0 = (I + \mathcal{A}_2(0))^{-1}(\Delta + 1)U_0$, and hence $\mathcal{A}_8(0)(\Delta + 1)U_0$ is the function g corresponding to the solution u of (5.1), (5.9) with $k = 0, f = (\Delta + 1)U_0$. Thus, the function $u + U_0$ solves (2.7), (2.8), that by the uniqueness completes the proof. \square

This lemma and the definition of \mathcal{A}_8 imply that

$$a(\mathcal{A}_8(k)(\Delta + 1)U_0) = c + k^2 h(k) \ln k,$$

where the function $h(k)$ is continuous w.r.t. small real k , and the same is true for $(kh(k))'$. We substitute this identity into (6.2), take into account (5.6) and lemmas 5.2 and 6.1, and arrive at

Lemma 6.2. *If the problem (2.7) has no bounded non-trivial solution satisfying (5.8) or (2.8), the operator $\mathcal{A}_1(k)(\mathbf{I} + \mathcal{A}_2(k))^{-1} \in \mathfrak{A}$ is bounded uniformly in small real k . If the problem (2.7) has the unique bounded non-trivial solution and it satisfies (2.8), the identity*

$$\begin{aligned} \mathcal{A}_1(k)(\mathbf{I} + \mathcal{A}_2(k))^{-1} &= (\ln k - \ln 2 + \mathbf{C})\mathbf{a}((\mathbf{I} + \mathcal{A}_2(0))^{-1}\cdot)\Psi \\ &+ \mathcal{A}_1(0)(\mathbf{I} + \mathcal{A}_2(0))^{-1} + k^2 \ln^3 k \mathcal{A}_9(k) \end{aligned}$$

holds true, where $\mathcal{A}_9 \in \mathfrak{A}$ is continuous w.r.t. small real k and the same is true for $(k\mathcal{A}_9(k))'$.

6.2. Presence of the unique logarithmically growing solution

In this subsection, we study the case when the problem (2.7) has the unique solution and this solution satisfies (5.9). We denote it by Ψ ; let $\phi \in L_2(\Pi_{\omega,\beta})$ be the associated solution to (5.4) with $k = 0$, $f = 0$.

We construct the solution to (5.4) as

$$g = \alpha\phi + \tilde{g}, \quad (6.3)$$

where $\alpha = \alpha(\tilde{g}, f)$ is a constant. We substitute this identity and (5.6) into (5.4) that yields

$$\begin{aligned} \tilde{g} + \mathcal{A}_2(0)\tilde{g} - (\ln k - \ln 2 + \mathbf{C})\mathbf{a}(\tilde{g})(\Delta + 1)U_0 + k^2 \ln k \mathcal{A}_7(k)\tilde{g} \\ - \alpha(\tilde{g}, f)(\ln k - \ln 2 + \mathbf{C})(\Delta + 1)U_0 + \alpha(\tilde{g}, f)k^2 \ln k \mathcal{A}_7(k)\phi = f. \end{aligned} \quad (6.4)$$

Here, we have used the identity

$$\mathbf{a}(\phi) = 1, \quad (6.5)$$

which follows from the definition of ϕ and (5.8), (5.9). Integrating by parts, one can check that

$$\int_{\Pi_{\omega,\beta}} \Psi(\Delta + 1)U_0 \, dx = -\gamma\pi^2.$$

Taking into account this formula and lemma 5.4, we calculate the inner product of (6.4) and Ψ ,

$$\begin{aligned} \gamma\pi^2(\ln k - \ln 2 + \mathbf{C})(\mathbf{a}(\tilde{g}) + \alpha(\tilde{g}, f)) + k^2 \ln k (\mathcal{A}_7(k)\tilde{g}, \Psi)_{L_2(\Pi_{\omega,\beta})} \\ + \alpha(\tilde{g}, f)k^2 \ln k (\mathcal{A}_7(k)\phi, \Psi)_{L_2(\Pi_{\omega,\beta})} = (f, \Psi)_{L_2(\Pi_{\omega,\beta})}. \end{aligned}$$

Hence,

$$\begin{aligned} \alpha(\tilde{g}, f) &= \frac{(f, \Psi)_{L_2(\Pi_{\omega,\beta})}}{\gamma\pi^2(\ln k - \ln 2 + \mathbf{C}) + k^2 \ln k (\mathcal{A}_7(k)\phi, \Psi)_{L_2(\Pi_{\omega,\beta})}} \\ &- \frac{\gamma\pi^2(\ln k - \ln 2 + \mathbf{C})\mathbf{a}(\tilde{g}) + k^2 \ln k (\mathcal{A}_7(k)\tilde{g}, \Psi)_{L_2(\Pi_{\omega,\beta})}}{\gamma\pi^2(\ln k - \ln 2 + \mathbf{C}) + k^2 \ln k (\mathcal{A}_7(k)\phi, \Psi)_{L_2(\Pi_{\omega,\beta})}}. \end{aligned} \quad (6.6)$$

We substitute this identity into (6.4) and obtain

$$\begin{aligned} \tilde{g} + \mathcal{A}_2(0)\tilde{g} + k^2 \ln k \mathcal{A}_{10}(k)\tilde{g} &= \tilde{f}, \\ \mathcal{A}_{10}(k) &:= \mathcal{A}_7(k) - \frac{\gamma\pi^2(\ln k - \ln 2 + \mathbf{C})\mathbf{a}(\cdot) + k^2 \ln k (\mathcal{A}_7(k)\cdot, \Psi)_{L_2(\Pi_{\omega,\beta})}}{\gamma\pi^2(\ln k - \ln 2 + \mathbf{C}) + k^2 \ln k (\mathcal{A}_7(k)\phi, \Psi)_{L_2(\Pi_{\omega,\beta})}} \mathcal{A}_7(k)\phi \\ &+ \frac{(\mathcal{A}_7(k)\cdot, \Psi)_{L_2(\Pi_{\omega,\beta})} - \mathbf{a}(\cdot)(\mathcal{A}_7(k)\phi, \Psi)_{L_2(\Pi_{\omega,\beta})}}{\gamma\pi^2(\ln k - \ln 2 + \mathbf{C}) + k^2 \ln k (\mathcal{A}_7(k)\phi, \Psi)_{L_2(\Pi_{\omega,\beta})}} (\ln k - \ln 2 + \mathbf{C})(\Delta + 1)U_0, \\ \tilde{f} &= f + \frac{(\ln k - \ln 2 + \mathbf{C})(\Delta + 1)U_0 - k^2 \ln k \mathcal{A}_7(k)\phi}{\gamma\pi^2(\ln k - \ln 2 + \mathbf{C}) + k^2 \ln k (\mathcal{A}_7(k)\phi, \Psi)_{L_2(\Pi_{\omega,\beta})}} (f, \Psi)_{L_2(\Pi_{\omega,\beta})}. \end{aligned}$$

It is easy to check that the operator \mathcal{A}_{10} is compact in $L_2(\Pi_{\omega,\beta})$ and bounded uniformly in small real k . One can also make sure that

$$(\tilde{f}, \Psi)_{L_2(\Pi_{\omega,\beta})} = 0, \quad (\mathcal{A}_{10}(k)\tilde{g}, \Psi)_{L_2(\Pi_{\omega,\beta})} = 0 \quad \text{for each } \tilde{g} \in L_2(\Pi_{\omega,\beta}).$$

Hence, $\tilde{f} \in \{\Psi\}^\perp$, where $\{\Psi\}^\perp$ is the orthogonal complement to Ψ in $L_2(\Pi_{\omega,\beta})$. Since the operator $(I + \mathcal{A}_2(0))^{-1} : \{\Psi\}^\perp \rightarrow \{\phi\}^\perp$ is bounded by lemma 5.4, we conclude that the operator $(I + \mathcal{A}_2(0) + k^2 \ln k \mathcal{A}_{10}(k))^{-1} : \{\Psi\}^\perp \rightarrow \{\phi\}^\perp$ is bounded uniformly in small real k . Thus,

$$\tilde{g} = (I + \mathcal{A}_2(0) + k^2 \ln k \mathcal{A}_{10}(k))^{-1} \tilde{f}.$$

We also note that by (6.5) and (6.6)

$$\begin{aligned} a(g) = \alpha(\tilde{g}) + a(\tilde{g}, f) &= \frac{(f, \Psi)_{L_2(\Pi_{\omega,\beta})}}{\gamma\pi^2(\ln k - \ln 2 + C) + k^2 \ln k (\mathcal{A}_7(k)\phi, \Psi)_{L_2(\Pi_{\omega,\beta})}} \\ &+ \frac{k^2 (\mathcal{A}_7(k)(a(\tilde{g})\phi \ln k - \tilde{g}), \Psi)_{L_2(\Pi_{\omega,\beta})}}{\gamma\pi^2(\ln k - \ln 2 + C) + k^2 \ln k (\mathcal{A}_7(k)\phi, \Psi)_{L_2(\Pi_{\omega,\beta})}}. \end{aligned}$$

These identities, the formulae for $\alpha(\tilde{g}, f)$ and \tilde{f} , (6.3) and lemma 5.2 lead us to

Lemma 6.3. *Suppose the problem (2.7) has the unique non-trivial solution and it satisfies (5.9). Then the operator $\mathcal{A}_1(k)(I + \mathcal{A}_2(k))^{-1} \in \mathfrak{A}$ is bounded uniformly in small real k .*

7. Eigenvalues emerging from the essential spectrum

This section is devoted to the proofs of theorems 2.5 and 2.6.

Proof of theorem 2.5. Given ω_ε , we describe the domain Π_{ω_ε} in terms of the Cartesian coordinates $\tilde{x} = (\tilde{x}', \tilde{x}_3)$, and introduce new variables as

$$x = \chi \begin{pmatrix} \tilde{r} \\ \tilde{\delta} \end{pmatrix} \mathcal{M}(\varepsilon)\tilde{x} + \left(1 - \chi \begin{pmatrix} \tilde{r} \\ \tilde{\delta} \end{pmatrix}\right) \tilde{x}, \tag{7.1}$$

where $\tilde{r} := \sqrt{\tilde{\tau}^2 + \tilde{x}_3^2}$, $(\tilde{\tau}, \tilde{s})$ are associated with \tilde{x}' and $\partial\omega$, $\tilde{\delta} > 0$ is small enough. The mapping $\mathcal{M}(\varepsilon)$ is described by the formulae

$$\tau = \tilde{\tau} - \varepsilon\beta(s), \quad \tilde{s} = s, \quad x_3 = \tilde{x}_3 \sqrt{1 + \frac{\varepsilon^2(\beta'(s))^2}{(1 - \tilde{\tau}k(s))^2}}.$$

It is clear that under this change of variables the domain Π_{ω_ε} is mapped onto Π_ω .

We rewrite the eigenvalue equation $\mathcal{H}_{\omega_\varepsilon}\psi = \lambda\psi$ in the variables x that leads us to

$$(\mathcal{H}_\omega - \varepsilon\mathcal{L}_4(\varepsilon))\psi = \lambda\psi, \tag{7.2}$$

where $\mathcal{L}_4(\varepsilon)$ is given by the expression on the right-hand side of (4.2) with the coefficients belonging to $C(T_\delta \times [0, \varepsilon_0])$, $\varepsilon_0 > 0$, and their supports lie inside T_δ . The operator \mathcal{L}_4 can be represented as

$$\begin{aligned} \mathcal{L}_4(\varepsilon) &= \mathcal{L}_5 + \varepsilon\mathcal{L}_6 + \varepsilon^2\mathcal{L}_7(\varepsilon), \quad \mathcal{L}_5 = \beta\chi \frac{\partial}{\partial\tau}(\Delta + 1) - (\Delta + 1)\beta\chi \frac{\partial}{\partial\tau}, \\ \mathcal{L}_6 &= \frac{1}{2}\beta^2 \frac{\partial}{\partial\tau} \chi^2 \frac{\partial}{\partial\tau}(\Delta + 1) + \frac{1}{2}(\Delta + 1)\beta^2 \chi^2 \frac{\partial^2}{\partial\tau^2} - \beta\chi \frac{\partial}{\partial\tau}(\Delta + 1)\beta\chi \frac{\partial}{\partial\tau} \\ &\quad - \frac{(\beta')^2 x_3 \chi}{2(1 - \tau k)^2} \frac{\partial}{\partial x_3}(\Delta + 1) + (\Delta + 1) \frac{(\beta')^2 x_3 \chi}{2(1 - \tau k)^2} \frac{\partial}{\partial x_3} \end{aligned} \tag{7.3}$$

where $\beta = \beta(s)$, $\chi = \chi(r/\tilde{\delta})$, and $\mathcal{L}_7(\varepsilon)$ is given by the expression on the right-hand side of (4.2) with the coefficients belonging to $C(T_{\tilde{\delta}} \times [0, \varepsilon_0])$, $\varepsilon_0 > 0$ and their supports lie inside $T_{\tilde{\delta}}$. The operators $\mathcal{L}_5, \mathcal{L}_6$ are in fact second-order differential operators satisfying (4.2) with compactly supported continuous coefficients. We write them in terms of Laplace operators since it is more convenient for the following arguments. We also observe that the operators $\mathcal{L}_i, i = 4, 5, 6$, are \mathcal{H}_ω -bounded by theorem 2.1 and the bounds can be estimated uniformly in ε .

We can rewrite (7.2) as

$$(\mathcal{H}_\omega - \lambda)\psi = \varepsilon \mathcal{L}_4 \psi. \tag{7.4}$$

If we denote now $\lambda = 1 - k^2$, we conclude that an eigenfunction ψ is a non-trivial solution to (5.1), (5.7) with $f = f_\varepsilon := \varepsilon \mathcal{L}_4 \psi$.

Let us find all values of k converging to zero as $\varepsilon \rightarrow +0$ for which the problem (5.1), (5.7) with $f = \varepsilon \mathcal{L}_4 \psi$ has a non-trivial solution. If the solution belongs to $\mathcal{D}(\mathcal{H}_\omega)$, it will imply that $\lambda = 1 - k^2$ is an eigenvalue of $\mathcal{H}_{\omega_\varepsilon}$ close to the threshold of the essential spectrum. In order to find such values, we employ the approach similar to that used in [BEG, B, G1, G2].

We note that in the case $d = \pi$ the eigenfunctions of $\mathcal{H}_{\omega_\varepsilon}$ are even w.r.t. x_3 that can be proved by analogy with [B, lemma 4.1]. Because of this in the case $d = \pi$ we restrict our considerations to even on x_3 functions.

It follows from (7.4) and lemma 5.1 that $\psi = \mathcal{A}_1(k)(I + \mathcal{A}_2(k))^{-1} f_\varepsilon$. We substitute this formula into (7.2) and obtain

$$f_\varepsilon - \varepsilon \mathcal{L}_4 \mathcal{A}_1(k)(I + \mathcal{A}_2(k))^{-1} f_\varepsilon = 0. \tag{7.5}$$

By the hypothesis and lemmas 2.4 and 5.3 in the case considered the problem (2.7), (5.8) can have at most one non-trivial solution and, if exists, it satisfies (5.9). Hence, by lemmas 6.2 and 6.3, the estimate (2.3) and the definition of \mathcal{L}_4 we conclude that the operator $\mathcal{L}_4 \mathcal{A}_1(k)(I + \mathcal{A}_2(k))^{-1}$ is bounded uniformly in ε and small real k as an operator in $L_2(\Pi_{\omega, \beta})$. Thus, for ε and small real k the operator $(I - \varepsilon \mathcal{L}_4 \mathcal{A}_1(k)(I + \mathcal{A}_2(k))^{-1})$ is boundedly invertible, and equation (7.5) has the trivial solution only. Therefore, equation (7.4) has no non-trivial solution for small ε and real k that completes the proof. \square

In the proof of theorem 2.6 we will employ

Lemma 7.1. *Suppose that there exists a non-trivial solution Ψ to (2.7) and (5.9). Then*

$$a(g) = -\frac{((I + \mathcal{A}_2(0))g, \Psi)_{L_2(\Pi_{\omega, \beta})}}{\gamma \pi^2}.$$

Proof. We denote $u := \mathcal{A}_1(0)g$. This function solves (5.1), (5.8) for $k = 0$, $f := (I + \mathcal{A}_2(0))g$. Now it is sufficient to integrate by parts in the integral

$$(f, \Psi)_{L_2(\Pi_{\omega, \beta})} = (f, \Psi)_{L_2(\Pi_\omega)} = -\int_{\Pi_\omega} \Psi(\Delta + 1)u \, dx$$

to prove the claimed formula. \square

Proof of theorem 2.6. We argue here as in the proof of theorem 2.5 up to equation (7.5). We substitute the representation for $\mathcal{A}_1(k)(I + \mathcal{A}_2(k))^{-1}$ given in lemma 6.2 into (7.5),

$$\begin{aligned} f_\varepsilon - \varepsilon(\ln k - \ln 2 + C)a(I + \mathcal{A}_2(0))^{-1} f_\varepsilon \mathcal{L}_4 \Psi \\ - \varepsilon \mathcal{L}_4 \mathcal{A}_1(0)(I + \mathcal{A}_2(0))^{-1} f_\varepsilon - \varepsilon k^2 \ln k \mathcal{L}_4 \mathcal{A}_9(k) f_\varepsilon = 0. \end{aligned} \tag{7.6}$$

By lemma 6.1 the operator $\mathcal{L}_4(\mathcal{A}_1(0)(\mathbf{I} + \mathcal{A}_2(0))^{-1} + k^2 \ln^3 k \mathcal{A}_9(k))$ is bounded uniformly in ε and small real k as an operator in $L_2(\Pi_{\omega, \beta})$. Hence, the operator

$$\mathbf{I} - \varepsilon \mathcal{L}_4(\mathcal{A}_1(0)(\mathbf{I} + \mathcal{A}_2(0))^{-1} + k^2 \ln^3 k \mathcal{A}_9(k))$$

is boundedly invertible. We denote the inverse by $\mathcal{A}_{11}(\varepsilon, k)$ and apply it to (7.6),

$$f_\varepsilon = \varepsilon \mathbf{a}((\mathbf{I} + \mathcal{A}_2(0))^{-1} f_\varepsilon) \mathcal{A}_{11}(\varepsilon, k) \mathcal{L}_4 \Psi. \tag{7.7}$$

We seek a non-trivial solution to (7.2). By lemma 5.1 it implies that the associated function f_ε is also non-trivial. Hence, by the identity (7.7), $\mathbf{a}((\mathbf{I} + \mathcal{A}_2(0))^{-1} f_\varepsilon) \neq 0$. Taking this inequality into account, we apply the functional $\mathbf{a}((\mathbf{I} + \mathcal{A}_2(0))^{-1} \cdot)$ to (7.7) and arrive at

$$1 = \varepsilon (\ln k - \ln 2 + \mathbf{C}) \mathbf{a}((\mathbf{I} + \mathcal{A}_2(0))^{-1} \mathcal{A}_{11}(\varepsilon, k) \mathcal{L}_4 \Psi). \tag{7.8}$$

The roots of this equation are values of k for which equation (7.5) has a non-trivial solution. This solution is unique up to a multiplicative constant and reads as follows:

$$f_\varepsilon = \varepsilon (\ln k - \ln 2 + \mathbf{C}) \mathcal{A}_{11}(\varepsilon, k) \mathcal{L}_4 \Psi. \tag{7.9}$$

The corresponding non-trivial solution to (5.1), (5.7) is given by $\psi_\varepsilon = \mathcal{A}_1(k)(\mathbf{I} + \mathcal{A}_2(k))^{-1} f_\varepsilon$. In view of (7.8) the coefficient \mathbf{c} in the asymptotics (5.7) satisfies the identity

$$\begin{aligned} \mathbf{c} &= -\frac{\sqrt{2\pi}}{4\sqrt{k}} \varepsilon \mathbf{a}((\mathbf{I} + \mathcal{A}_2(0))^{-1} \mathcal{A}_{11}(\varepsilon, k) \mathcal{L}_4 \Psi) + \mathcal{O}(1) \\ &= \frac{\sqrt{2\pi}}{4\sqrt{k}(\ln k - \ln 2 + \mathbf{C})} + \mathcal{O}(1), \quad k \rightarrow +0, \end{aligned}$$

where we have employed (7.8). Hence, \mathbf{c} is non-zero and the function ψ_ε decays at infinity (and thus is a needed eigenfunction), if and only if it is associated with a positive root to (7.8). We also note that for $k = 0$ equation (7.4) cannot have a non-trivial solution. Indeed, if so, it satisfies (5.8), that allows us to rewrite (7.4) as

$$f_\varepsilon - \varepsilon \mathcal{L}_4 \mathcal{A}_1(0)(\mathbf{I} + \mathcal{A}_2(0))^{-1} f_\varepsilon = 0.$$

By the boundedness of $\mathcal{A}_{11}(\varepsilon, 0)$ it implies $f_\varepsilon = 0$.

Let us study the existence of positive roots to (7.8). We rewrite this equation as

$$\frac{1}{\ln k - \ln 2 + \mathbf{C}} - \varepsilon \mathbf{a}((\mathbf{I} + \mathcal{A}_2(0))^{-1} \mathcal{A}_{11}(\varepsilon, k) \mathcal{L}_4 \Psi) = 0. \tag{7.10}$$

The properties of the operator \mathcal{A}_9 stated in lemma 6.2 and the definition of \mathcal{A}_{11} imply that the function on the left-hand side of this equation is real valued and continuous w.r.t. small non-negative k . Moreover,

$$\left| \frac{d}{dk} \mathbf{a}((\mathbf{I} + \mathcal{A}_2(0))^{-1} \mathcal{A}_{11}(\varepsilon, k) \mathcal{L}_4 \Psi) \right| \leq C,$$

where the constant C is independent of ε and small non-negative k . Hence, the derivation of the left-hand side in (7.10) is strictly negative for small positive k and small ε . Therefore, this equation has at most one positive root. It is clear that this root exists, if

$$\mathbf{a}((\mathbf{I} + \mathcal{A}_2(0))^{-1} \mathcal{A}_{11}(\varepsilon, 0) \mathcal{L}_4 \Psi) < 0, \tag{7.11}$$

and does not exist, if

$$\mathbf{a}((\mathbf{I} + \mathcal{A}_2(0))^{-1} \mathcal{A}_{11}(\varepsilon, 0) \mathcal{L}_4 \Psi) > 0. \tag{7.12}$$

By lemma 7.1 and the definition of \mathcal{A}_{11} we obtain that

$$\begin{aligned} \mathbf{a}((\mathbf{I} + \mathcal{A}_2(0))^{-1} \mathcal{A}_{11}(\varepsilon, 0) \mathcal{L}_4 \Psi) &= \mathbf{a}((\mathbf{I} + \mathcal{A}_2(0))^{-1} (\mathbf{I} + \varepsilon \mathcal{L}_4 \mathcal{A}_1(0) (\mathbf{I} + \mathcal{A}_2(0))^{-1}) \mathcal{L}_4 \Psi) + \mathcal{O}(\varepsilon^2) \\ &= C_0 + \varepsilon C_1 + \mathcal{O}(\varepsilon^2), \\ C_0 &:= \frac{(\mathcal{L}_5 \Psi, \Psi)_{L_2(\Pi_\omega)}}{\gamma \pi^2}, \quad C_1 := \frac{(\mathcal{L}_6 \Psi, \Psi)_{L_2(\Pi_\omega)} + (\mathcal{L}_5 \mathcal{A}_1(0) (\mathbf{I} + \mathcal{A}_2(0))^{-1} \mathcal{L}_5 \Psi, \Psi)_{L_2(\Pi_\omega)}}{\gamma \pi^2}. \end{aligned} \tag{7.13}$$

We denote $u := \mathcal{A}_1(0) (\mathbf{I} + \mathcal{A}_2(0))^{-1} \mathcal{L}_5 \Psi$. It follows from lemma 5.1 and (7.3) that u is the unique solution to (5.1), (5.8) for $k = 0$, $f = \mathcal{L}_5 \Psi = -(\Delta + 1) \beta \chi \frac{\partial \Psi}{\partial \tau}$. We denote now $\tilde{\Psi} := u - \beta \chi \frac{\partial \Psi}{\partial \tau}$ and conclude that there exists the unique solution to (2.7) satisfying (5.8) and (2.11). The first identity in (2.11) follows from the formula

$$\Psi(x) = l_\Psi(s) r^{1/2} \sin \frac{\theta}{2} + l_\Psi^{(1)}(s) r \sin \theta + l_\Psi^{(2)}(s) r^{3/2} \sin \frac{3\theta}{2} + \mathcal{O}(r^2), \quad r \rightarrow +0,$$

$l_\Psi^{(i)} \in C^\infty(\partial\omega)$, which can be proved by analogy with lemma 4.2 in [B]. Moreover, $l_\Psi \neq 0$, since otherwise the function $\frac{\partial \Psi}{\partial x_1} \in \dot{W}_2^1(\Pi_\omega)$ is a non-trivial solution to (2.7) belonging to $L_2(\Pi_\omega)$ that contradicts to the hypothesis. We also note that the function u satisfies (2.10) with l_Ψ replaced by $l_{\tilde{\Psi}}$.

We employ now all the aforementioned facts and (7.3), and integrate by parts,

$$\begin{aligned} C_0 &= -\frac{1}{\gamma \pi^2} \int_{\Pi_\omega} \Psi (\Delta + 1) \beta \chi \frac{\partial \Psi}{\partial \tau} dx = -i_1, \\ C_1 &= \frac{1}{2\gamma \pi^2} \int_{\Pi_\omega} \Psi (\Delta + 1) \beta^2 \chi^2 \frac{\partial^2 \Psi}{\partial \tau^2} dx - \frac{1}{\gamma \pi^2} \int_{\Pi_\omega} \Psi \beta \chi \frac{\partial}{\partial \tau} (\Delta + 1) \beta \chi \frac{\partial \Psi}{\partial \tau} dx \\ &\quad + \frac{1}{2\gamma \pi^2} \int_{\Pi_\omega} \Psi (\Delta + 1) \frac{(\beta')^2 x_3 \chi}{2(1 - \tau k)^2} \frac{\partial \Psi}{\partial x_3} dx + \frac{1}{\gamma \pi^2} \int_{\Pi_\omega} \Psi \beta \chi \frac{\partial}{\partial \tau} (\Delta + 1) u dx \\ &\quad - \frac{1}{\gamma \pi^2} \int_{\Pi_\omega} \Psi (\Delta + 1) \beta \chi \frac{\partial u}{\partial \tau} dx = -\frac{1}{\gamma \pi^2} \int_{\Pi_\omega} \Psi (\Delta + 1) \beta \chi \frac{\partial u}{\partial \tau} dx = -i_2. \end{aligned}$$

Hence, by (7.13),

$$\mathbf{a}((\mathbf{I} + \mathcal{A}_2(0))^{-1} \mathcal{A}_{11}(\varepsilon, 0) \mathcal{L}_4 \Psi) = -i_1 - \varepsilon i_2 + \mathcal{O}(\varepsilon^2). \tag{7.14}$$

It yields that the inequality (7.12) holds true, if the condition (2.15) is valid, i.e., in this case the operator $\mathcal{H}_{\omega_\varepsilon}$ has no eigenvalues converging to $1 - 0$ as $\varepsilon \rightarrow +0$. If the condition (2.12) is valid, it implies (7.11), and in this case the operator $\mathcal{H}_{\omega_\varepsilon}$ has the unique eigenvalue converging to $1 - 0$ as $\varepsilon \rightarrow +0$. This eigenvalue is given by $\lambda_\varepsilon = 1 - k_\varepsilon^2$, where k_ε is the root to (7.10). Formula (7.14) and equation (7.10) yield the asymptotics for k_ε ,

$$\begin{aligned} k_\varepsilon &= 2 e^{-C + \frac{i_2}{i_1}} e^{-\frac{1}{\varepsilon i_1}} (1 + \mathcal{O}(\varepsilon)), & \text{if } i_1 > 0, \\ k_\varepsilon &= e^{-\frac{1}{\varepsilon^2 i_2}} (C + \mathcal{O}(\varepsilon)), & \text{if } i_1 = 0, i_2 > 0, \end{aligned}$$

where C is a constant. These formulae prove (2.13).

The identities (7.8), (7.9) and the representation for $\mathcal{A}_1(k) (\mathbf{I} + \mathcal{A}_2(k))^{-1}$ given in lemma 6.2 imply that

$$\psi_\varepsilon(x) = \Psi(x) + \mathcal{O}(\varepsilon) \tag{7.15}$$

in $W_2^1(\Pi_{\omega, \tilde{\beta}})$ and $W_2^2(\Pi_{\omega, \tilde{\beta}} \setminus T_\delta)$ for each $\tilde{\beta} > 0$, $\delta > 0$. Given $\delta > 0$, we can choose $\tilde{\delta}$ in (7.1) small enough so that $\tilde{x} = x$ as $\tilde{x} \in \Pi_{\omega, \tilde{\beta}} \setminus T_{\tilde{\delta}}$. Hence, by (7.15) we conclude that

$$\psi_\varepsilon(x(\tilde{x})) = \Psi(\tilde{x}) + \mathcal{O}(\varepsilon)$$

in $W_2^2(S \setminus T_\delta)$ for each fixed bounded domain $S \subset \Pi_{\omega_\varepsilon}$ and each $\delta > 0$.

We pass to the variables \tilde{x} in (7.15), and in view of last identity we conclude that the asymptotic (2.14) is valid in the norm $W_2^1(\Pi_{\omega,\beta})$, if

$$\|\varphi_\varepsilon\|_{W_2^1(T_{\tilde{\delta}})} = \mathcal{O}(\sqrt{\varepsilon}), \quad \varphi_\varepsilon = \varphi_\varepsilon(\tilde{x}) := \Psi(x(\tilde{x})) - \Psi(\tilde{x}), \quad (7.16)$$

for a fixed $\tilde{\delta} > 0$ small enough. Here, the norm is understood in terms of variables \tilde{x} . The lowest eigenvalue of Dirichlet Laplacian in $T_{\tilde{\delta}}$ increases unboundedly as $\tilde{\delta} \rightarrow +0$. We employ this fact, the minimax principle and the obvious identity $\varphi_\varepsilon|_{\partial T_{\tilde{\delta}}} = 0$ to conclude that for $\tilde{\delta}$ small enough the inequality

$$\|\nabla\varphi_\varepsilon\|_{L_2(T_{\tilde{\delta}})}^2 \geq 2\|\varphi_\varepsilon\|_{L_2(T_{\tilde{\delta}})}^2$$

holds true. It is also clear that the $L_2(T_{\tilde{\delta}})$ -norm of φ_ε is bounded uniformly in ε , and $\|(\Delta_{\tilde{x}} + 1)\varphi_\varepsilon\|_{L_2(T_{\tilde{\delta}})} = \mathcal{O}(\varepsilon)$. We employ two last relations and integrate by parts,

$$\begin{aligned} \|\varphi_\varepsilon\|_{W_2^1(T_{\tilde{\delta}})}^2 &\leq 3(\|\nabla_{\tilde{x}}\varphi_\varepsilon\|_{L_2(T_{\tilde{\delta}})}^2 - \|\varphi_\varepsilon\|_{L_2(T_{\tilde{\delta}})}^2) \\ &= 3 \int_{\partial T_{\tilde{\delta}} \cap \{x_3=0\}} \varphi_\varepsilon \left[\frac{\partial\varphi_\varepsilon}{\partial\tilde{x}_3} \right] d\tilde{x}' - 3 \int_{T_{\tilde{\delta}}} \varphi_\varepsilon (\Delta_{\tilde{x}} + 1)\varphi_\varepsilon d\tilde{x} \\ &= 3 \int_{\partial\Pi_\omega \setminus \partial\Pi_{\omega_\varepsilon}} \Psi(x(\tilde{x})) \left[\frac{\partial\varphi_\varepsilon}{\partial\tilde{x}_3} \right] d\tilde{x}' - 3 \int_{\partial\Pi_{\omega_\varepsilon} \setminus \partial\Pi_\omega} \Psi(\tilde{x}) \left[\frac{\partial\varphi_\varepsilon}{\partial\tilde{x}_3} \right] d\tilde{x}' + \mathcal{O}(\varepsilon), \end{aligned}$$

where

$$\left[\frac{\partial\varphi_\varepsilon}{\partial\tilde{x}_3} \right] := \frac{\partial\varphi_\varepsilon}{\partial\tilde{x}_3}(\tilde{x}', -0) - \frac{\partial\varphi_\varepsilon}{\partial\tilde{x}_3}(\tilde{x}', +0).$$

It follows from (2.10) that the integrands in the remaining integrals are bounded uniformly in ε . Since the area of $\partial\Pi_\omega \setminus \partial\Pi_{\omega_\varepsilon}$ and $\partial\Pi_{\omega_\varepsilon} \setminus \partial\Pi_\omega$ is of order ε , we arrive at the identity (7.16).

The exponential decaying of ψ_ε at infinity is due to (5.7). □

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